The equivariant determinant of elliptic operators and the group action (Perspectives of Hyperbolic Spaces II)

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The equivariant determinant of elliptic operators and the group action.
楕円型作用素の同変行列式と群作用

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1. Equivariant determinant of elliptic operators

Let $M = M^{2m}$ be a $2m$-dimensional closed connected oriented Riemannian manifold and G a finite group acting on M. The G-action is assumed to be orientation-preserving, isometric and effective. Let $D : \Gamma(E) \to \Gamma(F)$ be a G-equivariant elliptic operator where E, F are complex G-vector bundles. Then $\ker D$ and $\operatorname{coker} D$ are finite dimensional G-modules.

Equivariant determinant of D is defined by

$$G \ni g \longrightarrow \det(D, g) = \frac{\det(g|\ker D)}{\det(g|\operatorname{coker} D)} \in S^1 \subset \mathbb{C}^*$$

and $\det_D := \det(D, \cdot\,) : G \longrightarrow S^1$ is a group homomorphism.

Then an additive group homomorphism $I_D : G \longrightarrow \mathbb{R}/\mathbb{Z}$ is defined by

$$I_D(g) := \frac{1}{2\pi\sqrt{-1}} \log\det(D, g) \pmod{\mathbb{Z}}.$$

This additive group homomorphism has the following properties:

$$I_D(gh) = I_D(hg) = I_D(g) + I_D(h)\ , \quad I_D([G, G]) = 0\ , \quad I_D(1) = 0\,.$$

Next proposition is proved by using the linear algebra (see [2]).

<u>**Proposition**</u> If $g^p = 1$ $(p \geq 2)$, we have

$$I_D(g) \equiv \frac{p-1}{2p}\mathrm{Ind}(D) - \frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{1-\xi_p^{-k}}\mathrm{Ind}(D, g^k) \pmod{\mathbb{Z}}$$

where $\xi_p = e^{2\pi\sqrt{-1}/p}$ is the primitive p-th root of unity,
$\mathrm{Ind}(D, g^k) = \mathrm{Tr}(g^k|\ker D) - \mathrm{Tr}(g^k|\mathrm{coker}\, D) \in \mathbb{C}$
is the equivariant index of D evaluated at g^k and
$\mathrm{Ind}(D) = \mathrm{Ind}(D, 1) = \dim\ker D - \dim\mathrm{coker}\, D \in \mathbb{Z}$
is the numerical index of D.

2. Cyclic action on Riemann surfaces and its rotation angles

Let Σ^σ be the compact Riemann surface of genus σ $(\sigma \geq 2)$. Assume that a finite group G acts on Σ^σ as a biholomorphic automorphism with respect to some complex structure of Σ^σ.

Let $g \in G$ be any element of order p and set $\mathbb{Z}_p = \langle g\rangle$. Then $\pi : \Sigma^\sigma \longrightarrow \Sigma^\sigma/\mathbb{Z}_p$ is a branched covering with b branch points $y_1, \cdots, y_b \in \Sigma^\sigma/\mathbb{Z}_p$ of order $(n_1, \cdots, n_b)$, where $\pi^{-1}(y_i) = \{q_i, g\cdot q_i, \cdots, g^{r_i-1}\cdot q_i\}$ consists of $r_i := p/n_i$ points.

For $1 \leq i \leq b$, assume that $g^{r_i}|T_{\pi^{-1}(y_i)}\Sigma^\sigma = \xi_{n_i}^{t_i} = \xi_p^{r_i t_i}$ where $1 \leq t_i \leq n_i - 1$ and t_i is prime to n_i

Problem 1 Can we determine the rotation angles $r_1t_1, \cdots, r_bt_b$ by using the equivariant determinant ?

Let D_ℓ be the $\otimes^\ell T\Sigma^\sigma$-valued Dirac operator on Σ^σ defined by the complex structure of Σ^σ. Then using the Atiyah-Singer index formula, we can show the next formula (see [2]).

<u>**Formula**</u> Set

$$\Phi_i := zt_i(n_i-1)(7n_i-11) + 6\sum_{j=\left[\frac{(\ell+1)zt_i}{n_i}\right]+1}^{\left[\frac{(\ell+n_i+1)zt_i}{n_i}\right]} f_{n_i}\left(\left[\frac{jn_i-1}{zt_i}\right] - \ell - 1\right)$$

where $f_{n_i}(x) = x^2 - (n_i - 2)x - (n_i - 1)^2$ and [ ] is the Gauss's symbol. Then for any integers ℓ, z, $12p\, I_{D_\ell}(g^z)$ is an integer and we have

$$12p\, I_{D_\ell}(g^z) \equiv 6(p-1)(1-\sigma)(2\ell+1) + \sum_{i=1}^{b} r_i \Phi_i \pmod{12p}\,.$$

<u>**Remark**</u> Assume that $\mu\nu$ is prime to p. Then since $p\, I_{D_\ell(g)} = 0$,
$\mu\, I_{D_\ell}(g^\nu) = \mu\nu\, I_{D_\ell}(g) = 0 \iff I_{D_\ell}(g) = 0$.

<u>**Example 1**</u> (Dihedral group) Assume that p is odd. Let

$$G = D(p) = \langle g,\, h \,|\, g^p = h^2 = 1,\; g^{-1}h^{-1}gh = g^{-2} \rangle$$

be the dihedral group. Then since $g^{-2} \in [G, G]$, it follows that

$$I_{D_\ell}(g^{-2}) = -2I_{D_\ell}(g) = 0 \;\; (\forall \ell \in \mathbb{N}) \iff I_{D_\ell}(g) = 0 \;\; (\forall \ell \in \mathbb{N})\,.$$

<u>**Example 2**</u> (Symmetric group) Assume that p is odd. Let $\tau_1 = (1,2)$, $\tau_2 = (1,3)$, $\cdots$, $\tau_{p-1} = (1,p)$ be the transpositions of p letters and $S(p)$ the symmetric group of the p letters. Let $g \in S(p)$ be an element of order p defined by $g = \tau_1\tau_2\cdots\tau_{p-1} = (p, p-1, \cdots, 2, 1)$. Then we have

$$\begin{aligned} 0 &= I_{D_\ell}(1) = I_{D_\ell}((g\tau_{p-1}\cdots\tau_2\tau_1)^2) \\ &= I_{D_\ell}(g^2) + I_{D_\ell}(\tau_{p-1}^2) + \cdots + I_{D_\ell}(\tau_1^2) = 2I_{D_\ell}(g) \\ &\iff I_{D_\ell}(g) = 0 \;\; (\forall \ell \in \mathbb{N})\,. \end{aligned}$$

$\boxed{\textbf{Problem 2}}$ Can we determine the rotation angles $r_1t_1, \cdots, r_bt_b$ of g under the condition that $I_{D_\ell}(g) = 0$ for any integers ℓ ?

Assume that the order p of g is an odd prime number hereafter.
(Hence we have $n_i = p$, $r_i = 1$ for $1 \le i \le b$.)
Then the we have the following formula.

<u>**Formula**</u> (Riemann-Hurwitz equation)

$$\sigma = p(\tau - 1) + \frac{b(p-1)}{2} + 1 \iff \tau = \frac{1}{p}\left(\sigma - \frac{b(p-1)}{2} - 1\right) + 1$$

where τ is the genus of $\Sigma^\sigma/\mathbb{Z}_p$.

Let $F := \{q_1, \cdots, q_b\} \subset \Sigma^\sigma$ be the fixed point set of the $\mathbb{Z}_p$-action and $\pi : \Sigma^\sigma \longrightarrow \Sigma^\tau = \Sigma^\sigma/\mathbb{Z}_p$ the branched covering with branch points $\pi(q_1), \cdots, \pi(q_b)$ of order $(p, \cdots, p)$.

Assume that $g|T_{q_i}\Sigma^\sigma = \xi_p^{t_i}$ $(1 \le t_i \le p-1)$ for $1 \le i \le b$.

Set $\Sigma_0^\sigma := \Sigma_0^\sigma - F$ and $\Sigma_0^\tau := \Sigma_0^\sigma/\mathbb{Z}_p$, then we have the next exact sequence:

$$\pi_1(\Sigma_0^\sigma) \longrightarrow$$

$$\pi_1(\Sigma_0^\tau) = \left\langle a_1, b_1, \cdots, a_\tau, b_\tau, x_1, \cdots, x_b \;\middle|\; \prod_{k=1}^{\tau} [a_k, b_k] x_1 \cdots x_b = 1 \right\rangle$$

$$\xrightarrow{\partial} \mathbb{Z}_p \longrightarrow 0$$

where x_i is represented by a loop around the branch point $\pi(q_i)$. Let $\bar{t}_i$ denotes the mod.p inverse of t_i $(1 \le i \le b)$. Then since

$$\prod_{k=1}^{\tau} [a_k, b_k] x_1 \cdots x_b = 1 \,, \quad \partial([a_k, b_k]) = 0 \,, \quad \partial(x_i) = \bar{t}_i \in \mathbb{Z}_p \,,$$

it follows that

$$\sum_i \partial(x_i) = 0 \in \mathbb{Z}_p \iff \sum_{i=1}^{b} \bar{t}_i \equiv 0 \pmod{p} \cdots (1)$$

Conversely, if $b \ge 2$, σ, τ satisfy the Riemann-Hurwitz equation and the equality (1) holds, then there exists a $\mathbb{Z}_p$-action on Σ^σ with b-fixed points such that the genus of $\Sigma^\sigma/\mathbb{Z}_p$ is τ and that the rotation angles are $t_1, \cdots, t_b$ (see [1]).

By definition, rotation angles $(t_1, \cdots, t_b)$ are equivalent to the rotation angles $(t'_1, \cdots, t'_b)$ if there exists an integer s such that $t'_i = st_i$ $(\forall i)$ or $(t'_1, \cdots, t'_b)$ is a permutation of $(t_1, \cdots, t_b)$.

In the following tables, the equivalence class of rotation angles of g is simply called " rotation angles ", the rotation angles of g such that $\Sigma_{i=1}^b \bar{t}_i \equiv 0$ (mod. p) is called " possible rotation angles " and the rotation angles of g such that $I_{D_\ell}(g) = 0$ $(\forall \ell \in \mathbb{N})$ is called " admissible rotation angles ".

$p = 5$

σ	b	Possible rotation angles	Admissible rotation angles
2	3	$(1,1,2)$	none
3	1	none	none
4	4	$(1,1,1,3)$ $(1,1,4,4)$ $(1,2,3,4)$	$(1,1,4,4)$ $(1,2,3,4)$
5	2	$(1,4)$	$(1,4)$
6	0		
	5	$(1,1,1,1,1)$ $(1,1,1,2,4)$ $(1,1,2,2,3)$	$(1,1,1,1,1)$
7	3	$(1,1,2)$	none
8	6	$(1,1,1,1,2,2)$ $(1,1,1,1,3,4)$ $(1,1,1,2,3,3)$ $(1,1,1,4,4,4)$ $(1,1,2,3,4,4)$	$(1,1,1,4,4,4)$ $(1,1,2,3,4,4)$
9	4	$(1,1,1,3)$ $(1,1,4,4)$ $(1,2,3,4)$	$(1,1,4,4)$ $(1,2,3,4)$
10	2	$(1,4)$	$(1,4)$
	7	$(1,1,1,1,1,1,4)$ $(1,1,1,1,1,2,3)$ $(1,1,1,1,2,4,4)$ $(1,1,1,1,3,3,3)$ $(1,1,1,2,2,3,4)$ $(1,1,1,3,3,4,4)$	$(1,1,1,1,1,1,4)$ $(1,1,1,1,1,2,3)$
11	0		
	5	$(1,1,1,1,1)$ $(1,1,1,2,4)$ $(1,1,2,2,3)$	$(1,1,1,1,1)$

$\underline{p = 7}$

σ	b	Possible rotation angles	Admissible rotation angles
2	none	none	none
3	3	$(1,1,3)$ $(1,2,4)$	$(1,2,4)$
4	1	none	none
5	none	none	none
6	4	$(1,1,1,2)$ $(1,1,4,5)$ $(1,1,6,6)$ $(1,2,5,6)$	$(1,1,6,6)$ $(1,2,5,6)$
7	2	$(1,6)$	$(1,6)$
8	0	(free action)	
9	5	$(1,1,1,1,5)$ $(1,1,1,3,6)$ $(1,1,1,4,4)$ $(1,1,2,3,5)$ $(1,1,2,4,6)$ $(1,1,3,3,4)$	$(1,1,2,4,6)$
10	3	$(1,1,3)$ $(1,2,4)$	$(1,2,4)$
11	1	none	none

$p = 11$

σ	b	Possible rotation angles	Admissible rotation angles
2	none	none	none
3	none	none	none
4	none	none	none
5	3	$(1,1,5)$ $(1,2,3)$	none
6	1	none	none
7	none	none	none
8	none	none	none
9	none	none	none
10	4	$(1,1,1,7)$ $(1,1,2,4)$ $(1,1,3,9)$ $(1,1,6,8)$ $(1,1,10,10)$ $(1,2,7,8)$ $(1,2,9,10)$ $(1,3,8,10)$	$(1,1,10,10)$ $(1,2,9,10)$ $(1,3,8,10)$
11	2	$(1,10)$	$(1,10)$

3. Higher dimensional case

Let M be a $2m$-dimensional almost complex manifold with $\mathbb{Z}_p$-action and q_i $(1 \le i \le b)$ the fixed points of the generator g of $\mathbb{Z}_p$. Then for $1 \le i \le b$, the tangent space $T_{q_i}M$ is decomposed into

$$T_{q_i}M = \bigoplus_{j=1}^{m} V_{i,j} \quad \left(\dim_{\mathbb{C}} V_{i,j} = 1\ , \quad g|V_{i,j} = \xi_p^{t_{i,j}}\right) .$$

We call $(\{t_{1,1}, \cdots , t_{1,m}\}, \cdots , \{t_{b,1}, \cdots , t_{b,m}\})$ the rotation angles of g.

Example 3 Let $D(5) = \langle g, h \,|\, g^5 = h^2 = 1,\ g^{-1}h^{-1}gh = g^{-2}\rangle$ be the dihedral group. Then since Σ^5 can be embedded symmetrically into $\mathbb{R}^3$ with respect to the π-rotation around x-axis and $2\pi/5$-rotation around z-axis, $D(5)$ can act on Σ^5 and g acts on Σ^5 with 2-fixed points of the rotation angles $(1,4)$, $(2,3)$. Hence the diagonal action of $D(5)$ on $\Sigma^5 \times \Sigma^5$ gives an action of g on $\Sigma^5 \times \Sigma^5$ with 4-fixed points of the rotation angles

$$(1,4) \times (1,4) = (\{1,1\},\{1,4\},\{1,4\},\{4,4\}),$$
$$(1,4) \times (2,3) = (\{1,2\},\{1,3\},\{2,4\},\{3,4\})$$

and we have $I_D(g) = 0 \in \mathbb{Z}_5$ for any $D(5)$-equivariant elliptic operator D because $-2I_D(g) = I_D(g^{-2}) = I_D(g^{-1}h^{-1}gh) = 0$.

Now assume that $\mathbb{Z}_5 = \langle g \rangle$ acts on $\Sigma^5 \times \Sigma^5$ and that the action preserves some almost complex structure of $\Sigma^5 \times \Sigma^5$. Let L be the complex $\mathbb{Z}_5$-line bundle defined by

$$L = \left(\wedge^2_{\mathbb{C}} T(\Sigma^5 \times \Sigma^5)\right)^{\ell}$$

and D_ℓ the L-valued Dirac operator on $\Sigma^5 \times \Sigma^5$.

$\boxed{\textbf{Problem 3}}$ Can we determine the rotation angles of g under the condition that g has 4-fixed points and $I_{D_\ell}(g) = 0 \in \mathbb{Z}_5$ for any integers ℓ ?

Let $(\{s_1,t_1\},\{s_2,t_2\},\{s_3,t_3\},\{s_4,t_4\})$ be the rotation angles of g. Then using the Atiyah-Singer index formula, we can prove the next equality.

$$I_{D_\ell}(g) = \frac{32}{5}(2\ell+1)^2 - \frac{1}{5}\sum_{i=1}^{4}\sum_{k=1}^{4} \frac{\xi_5^{k\ell(s_i+t_i)}}{(1-\xi_5^{-k})(1-\xi_5^{-ks_i})(1-\xi_5^{-kt_i})}$$

Equivalence of rotation angles is defined as follows:

$$(\{1,2\},\{1,2\},\{2,3\},\{3,4\}) \equiv (\{3,4\},\{2,1\},\{3,2\},\{1,2\})$$
$$\equiv (\{2,4\},\{2,4\},\{4,1\},\{1,3\}) \equiv (\{3,1\},\{3,1\},\{1,4\},\{4,2\}) \equiv \cdots$$

Then we can obtain the following result.

Result The (equivalence class of) rotation angles do not satisfy the condition $I_{D_\ell}(g) = 0$ $(\forall \ell)$ unless

$$(\{1,1\},\{1,4\},\{1,4\},\{4,4\}), \quad (\{1,1\},\{2,3\},\{1,4\},\{4,4\}),$$
$$(\{1,1\},\{2,3\},\{2,3\},\{4,4\}), \quad (\{1,2\},\{1,2\},\{3,4\},\{3,4\}),$$
$$(\{1,2\},\{1,3\},\{2,4\},\{3,4\})$$

(see Example 3).

Remark Let N be the number of the equivalence classes of rotation angles. Then we have

$$N \geq \frac{4^8}{2^4 \times 4! \times 4} = \frac{128}{3} \implies N \geq 43$$

References

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